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Logarithmic Primary Fields in Conformal and Superconformal Field Theory

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Abstract

In this note, some aspects of the generalization of a primary field to the logarithmic scenario are discussed. This involves understanding how to build Jordan blocks into the geometric definition of a primary field of a conformal field theory. The construction is extended to $N = 1, 2$ superconformal theories. For the $N = 0, 2$ theories, the two-point functions are calculated.

1 Introduction

In recent years, Logarithmic Conformal Field Theories (LCFTs) have come under much investigation[1]. The reasons include possible applications to statistical physics [12], possible applications to WZW theories [4][11], possible applications to D-Brane dynamics [7][8], and a potential understanding of how to control non-unitary quantum field theories. LCFTs are characterized by non-unitary behaviour, such as logarithms in correlation functions and indecomposable Jordan blocks in the Virasoro representation theory.

Even though a LCFT has such recognizable characteristics, a clear cut definition of a LCFT still does not really exist. Moreover, how the usual machinery of CFT generalizes to the LCFT case is still not completely understood. Many approaches to these problems have been followed [2][5]. In this note, the construction of a LCFT in terms of its primary fields will be analyzed, following [13][6][9]. This approach introduces indecomposable Jordan blocks by hand, and essentially comes down to modifying existing constructions by introducing nilpotent 'variables', which algebraically mimic the Jordan block structure.

In this note, many of the features of the bosonic are reviewed in a slightly different light from the previous literature. In particular, a more geometric approach is considered, and instead of nilpotent variables, Jordan blocks are used from the outset. This approach is then generalized to the supersymmetric $N = 1, 2$ cases.

Sections 2-4 describe the bosonic theory in a manner that naturally generalizes to the supersymmetric case. In these sections, a primary field is defined, and the infinitesimal transformations obtained. In order to verify the differential operators obtained indeed give the required primary field, the generators of the infinitesimal transformation are exponentiated. As an application, using global conformal symmetry, the two point function is calculated. These sections yield and extend some results from [13][9][3].

Using the machinery developed in sections 2-4, section 5 looks at the $N = 1$ theory, obtaining and extending some results from [6], although in a more geometric fashion. Since the bosonic part of the Cartan sub-algebra of the $N = 1$ theory is same as for the bosonic case, the machonery works in much the same way.

Sections 6-8 look at the $N = 2$ theory. Since the Cartan subalgebra is larger than the $N = 0, 1$ theories, more Jordan blocks can potentially appear. Section 7 defines a $N = 2$ logarithmic primary field that accounts for these extra Jordan Blocks. In order to study what further logarithms might occur, the two point function is calculated using global conformal symmetry.

2 General Framework

Consider the one-form dz, the matrix

$$M = \begin{pmatrix} h & 1 \\ 0 & h \end{pmatrix} = h\mathbb{I}_2 + J \tag{1}$$

and the formal one-form dz^M. Then, using $a^b = \exp(b \log a)$ and the series definition for exp,

$$dz^M = dz^{h\mathbb{I}_2 + J} = dz^{h\mathbb{I}_2} dz^J = \mathbb{I}_2 dz^h \exp(J \log(dz)) = dz^h(\mathbb{I}_2 + J\log(dz)) \tag{2}$$

A more proper definition of dz^M might be $dz^h(\mathbb{I}_2 + J\log(dz))$, although exactly what $\log(dz)$ means is not apparent to the author. Let the matrix act on a column vector

$$v = \begin{pmatrix} \phi_0(z) \\ \phi_1(z) \end{pmatrix} \tag{3}$$

and consider a conformal transformation $f : z \mapsto z'$. Then, under pull-back, one has for $dz^M v$

$$f^*\left(dz'^h \begin{pmatrix} 1 & \log(dz') \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_0(z') \\ \phi_1(z') \end{pmatrix} \right) \tag{4}$$

$$= \left(\frac{dz'}{dz}\right)^h dz^h \begin{pmatrix} 1 & \log\left(\frac{dz'}{dz}\right) + \log(dz) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (\phi_0 \circ f)(z) \\ (\phi_1 \circ f)(z) \end{pmatrix} \tag{5}$$

$$= \left(\frac{dz'}{dz}\right)^h dz^h \begin{pmatrix} \phi_0 + \log\left(\frac{dz'}{dz}\right)\phi_1 + \log(dz)\phi_1 \\ \phi_1 \end{pmatrix} \tag{6}$$

$$=: dz^h \begin{pmatrix} \phi_0'(z) + \log(dz)\phi_1'(z) \\ \phi_1'(z) \end{pmatrix} \tag{7}$$

This then gives the well known transformations for a logarithmic primary field [13]. It seems that $\log(dz)$, is well-defined in the algebraic sense, up to the arbitrary phase that can be added, although the notion of constructing a geometric object out of $\log(dz)$ seems very unusual. Here, it has been assumed that dz and $\log(dz)$ are linearly independent.

More generally, one could consider raising dz to the power $h + J$, where $J^n = 0$, $J^{n-1} \neq 0$, $hJ = Jh$, and h not nilpotent, which has a unique (up to conjugation), faithful representation of smallest dimension $= n$. Choosing the n-dimensional representation where J has 1s just above the leading diagonal, and is zero elsewhere, and going through the same machinery, one finds

$$dz^{h+J} = dz^h \sum_{i=0}^{n-1} \frac{J^i (\log dz)^i}{i!} \tag{8}$$

Now, v is a column

$$v_i = \phi_i(z), \qquad i = 0, \ldots, n-1 \tag{9}$$

and hence

$$(dz^{h+J} v)_i = dz^h \sum_{j=0}^{n-i-1} \frac{1}{j!} \phi_{j+i}(z) (\log dz)^j \tag{10}$$

Pulling back gives

$$f^*(dz'^{h+J}v)_i(z) \;=\; \left(\frac{dz'}{dz}\right)^h dz^h \sum_{j=0}^{n-i-1} \frac{1}{j!}(\phi_{j+i}\circ f)(z)\Big(\log\left(\frac{dz'}{dz}\right)+\log dz\Big)^j \tag{11}$$

$$=: \; dz^h \sum_{j=0}^{n-i-1} \frac{1}{j!}\phi'_{j+i}(z)(\log dz)^j \tag{12}$$

Looking at the $(\log dz)^l$ term yields

$$\phi'_{l+i}(z) = \left(\frac{dz'}{dz}\right)^h \sum_{j=0}^{n-1-i-l} \frac{1}{j!}(\phi_{j+i+l}\circ f)(z)\Big(\log\left(\frac{dz'}{dz}\right)\Big)^j \tag{13}$$

which only depends on $i+l$, rather than i and l independently. One thus obtains the transformation law

$$\phi'_i(z) = \left(\frac{dz'}{dz}\right)^h \sum_{j=0}^{n-1-i} \frac{1}{j!}(\phi_{j+i}\circ f)(z)\Big(\log\left(\frac{dz'}{dz}\right)\Big)^j \tag{14}$$

which are just the components of

$$v'(z) = \left(\frac{dz'}{dz}\right)^{h+J}(v\circ f)(z) \tag{15}$$

as might be expected from (2). Considering $z' = z + az^{n+1}$, for a infinitesimal, leads to

$$\delta\phi_i(z) := \phi'_i(z) - \phi_i(z) = a\Big(h(n+1)z^n\phi_i + z^{n+1}\partial\phi_i + (n+1)z^n\phi_{i+1}\Big) \tag{16}$$

This is the well known infinitesimal transformation law for a logarithmic primary field [13]. These transformations give rise to the $n\times n$-matrix valued vector fields

$$l_n = \mathbb{I}(h(n+1)z^n + z^{n+1}\partial) + J(n+1)z^n \tag{17}$$

which act on v, and can be readily verified to satisfy the Witt algebra. Just because the infinitesimal form matches up, does not necessarily imply that (16) integrates up to (14) by exponentiation. This must be checked explicitly.

3 Exponentiation

What must be checked is that

$$\exp(al_n)\phi_i(z) = \phi'_i(z) \tag{18}$$

Since a closed form has been conjectured, this can be checked inductively on the order of a. The inductive step going from the a^q to the a^{q+1} is by acting on the a^q term with $\frac{a}{q+1}l_n$. The most calculationally instructive way is to build up to this from the simplest

case. Consider the case of just the co-ordinate transformation, $z' \mapsto z$, and $l_n = z^{n+1}\frac{d}{dz}$. Then

$$z' = z(1-naz^n)^{-\frac{1}{n}} = z + \sum_{q=1}^{\infty} \frac{(1)(1+n)\dots(1+n(q-1))}{q!} a^q z^{nq+1} = z + \delta z \tag{19}$$

can be checked inductively to show that

$$\exp(al_n)z = z(1-naz^n)^{-\frac{1}{n}} \tag{20}$$

for $n \neq 0$ (the $n=0$ case is omitted throughout, which just corresponds to a dilation). Next, consider a function under pull-back g^*, so that

$$f'(z) = (f \circ g)(z) = f(z(1-naz^n)^{-\frac{1}{n}}) = f(z+\delta z) \tag{21}$$

Now Taylor expand in δz, with the expression for δz given by (19). The a^q term is then given by

$$f'(z)|_{a^q} = \frac{a^q z^{nq}}{q!} \sum_{k=1}^{q} \frac{(-1)^k}{k!} \sum_{p=1}^{k} \left(\begin{array}{c} p \\ k \end{array} \right) p(p+n)\dots(p+n(q-1)) z^k \partial^k \phi \tag{22}$$

In order to get the $a^{q+1}\partial^{k+1}\phi$ term, one must act on the $a^q\partial^k\phi$ and $a^q\partial^{k+1}\phi$ terms with $\frac{1}{q+1}z^{n+1}\partial$, and indeed the induction follows through. For a primary field, a similar procedure can be used, where there is now a multiplicative factor of $(\frac{dz'}{dz})^h$. Using (19), one finds

$$\left(\frac{dz'}{dz}\right)^h = 1 + \sum_{p=1}^{\infty} \frac{1}{p!} h(n+1)(h(n+1)+n)\dots(h(n+1)+n(p-1)) a^p z^{np} \tag{23}$$

from which the $a^r\partial^k\phi$ term can be deduced, yielding

$$\begin{aligned}
\phi'(z)|_{a^r\partial^k\phi} \quad = \quad & \frac{a^r z^{nr}}{r!}(-1)^k \sum_{s=1}^{k} \frac{(-1)^s}{(k-s)!s!} s(s+n)\dots(s+n(r-1)) z^k \partial^k \phi + \\
& \frac{z^{nr}}{r!}(-1)^k h(n+1)\dots(h(n+1)+n(p-1)) \sum_{s=0}^{k} \frac{(-1)^s}{(k-s)!s!} z^k \partial^k \phi + \\
& \sum_{p+q=r;\ p,q\geq 1} \frac{a^r z^{nr}}{p!q!} h(n+1)\dots(h(n+1)+n(p-1)) \times \\
& \quad (-1)^k \sum_{s=1}^{k} \frac{(-1)^s}{(k-s)!s!} s(s+n)\dots(s+n(q-1)) z^k \partial^k \phi
\end{aligned} \tag{24}$$

This can the be used in exactly the same manner as the case of the function, with

$$l_n = h(n+1)z^n + z^{n+1}\partial \tag{25}$$

The induction is a little more involved, calculationally, than the case of the function, but follows through in very similar manner. Before moving to the case of the logarithmic primary field, a way of dealing with powers of log must be found. The conjecture

$$(-\log(1-\lambda))^k = k!\sum_{p_k=k}^{\infty}\sum_{p_{k-1}=k-1}^{p_k-1}\dots\sum_{p_1=1}^{p_2-1}\frac{1}{p_kp_{k-1}\dots p_1}\lambda^{p_k} \tag{26}$$

must be verified, which is easily done by induction, noting that

$$\frac{d}{d\lambda}\Big((-\log(1-\lambda))^k\Big) = k(-\log(1-\lambda))^{k-1}(1-\lambda)^{-1} = k(-\log(1-\lambda))^{k-1}\sum_{j=0}^{\infty}\lambda^j \tag{27}$$

The integration constant is fixed by noting that $(\log(1-\lambda))^k$ has leading term λ^k. Hence, given z' in (19),

$$\left(\log\left(\frac{dz'}{dz}\right)\right)^k = \left(\frac{n+1}{n}\right)^k k!\sum_{p_k=k}^{\infty}\sum_{p_{k-1}=k-1}^{p_k-1}\dots\sum_{p_1=1}^{p_2-1}\frac{1}{p_kp_{k-1}\dots p_1}(naz^n)^{p_k} \tag{28}$$

For the logarithmic field, it suffices to consider the $a^v\partial^r\phi_{k+i}$ term, where v, r, k are fixed, and induction performed on them. Then,

$$\begin{aligned}\phi'(z)|_{a^v\partial^r\phi_{k+i}} = \left(\frac{n+1}{n}\right)^k\Bigg[&\sum_{u_{k-1}=k-1}^{v-1}\sum_{u_{k-2}=k-2}^{u_{k-1}-1}\dots\sum_{u_1=1}^{u_2-1}\frac{n^v}{vu_{k-1}\dots u_1}\sum_{s=0}^{r}\frac{(-1)^{s+r}}{s!(r-s)!} + \\ &\Bigg(\sum_{p=1}^{v-k}\sum_{u_{k-1}=k-1}^{v-p-1}\sum_{u_{k-2}=k-2}^{u_{k-1}-1}\dots\sum_{u_1=1}^{u_2-1}\frac{1}{(v-p)u_{k-1}\dots u_1}\frac{n^{v-p}}{p!}\times \\ &h(n+1)\dots(h(n+1)+n(p-1))\sum_{s=0}^{r}\frac{(-1)^{s+r}}{s!(r-s)!}\Bigg) + \\ &\Bigg(\sum_{q=1}^{v-k}\sum_{u_{k-1}=k-1}^{v-q-1}\sum_{u_{k-2}=k-2}^{u_k}\dots\sum_{u_1=1}^{u_2-1}\frac{1}{(v-q)u_k\dots u_1}\frac{n^{v-q}}{q!}\times \\ &\sum_{s=1}^{r}\frac{(-1)^{r+s}}{s!(r-s)!}s(s+n)\dots(s+n(q-1))\Bigg) + \\ \Bigg(\sum_{t=2}^{v-k}\sum_{u_{k-1}=k-1}^{v-t-1}\sum_{u_{k-2}=k-2}^{u_{k-1}-1}&\dots\sum_{u_1=1}^{u_2-1}\sum_{q=1}^{t-1}\frac{1}{(v-t)u_{k-1}\dots u_1}\frac{n^{v-t}}{q!(t-q)!}h(n+1)\dots \\ (h(n+1)+n(t-q-1))\sum_{s=1}^{r}&\frac{(-1)^{s+r}}{s!(r-s)!}s(s+n)\dots(s+n(q-1))\Bigg)\Bigg]z^{nv+r}\partial^r\phi_{k+i}\end{aligned} \tag{29}$$

For the induction, the vector field l_n now takes the form (17). In order to find the $a^{v+1}\partial^{r+1}\phi_{k+i+1}$ term, the $a^v\partial^{r+1}\phi_{k+i+1}$, $a^v\partial^r\phi_{k+i+1}$ and $a^v\partial^{r+1}\phi_{k+i}$ terms must be considered. The induction is very messy and tedious, but follows through in a very similar manner to the previous cases. Thus, for a logarithmic field, (18) is verified, by induction. Note that nothing about whether or not ϕ has a Laurent expansion has been assumed, only that in a suitably small neighbourhood, it is possible to Taylor expand ϕ.

4 Two Point Function

So far, the fields ϕ_i have been represented by a vector. However, as will be more useful in the following, they can be represented as a matrix. In this instance, moving from a vector to a matrix is analagous to moving from a vector bundle to its associated G-bundle. In this sense, once given the 'one-form' (8), which generates the transformation laws, the description of sections as matrices or vectors is equivalent. In the rank 2 case this looks like

$$\begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} \mapsto \begin{pmatrix} \phi_1 & \phi_0 \\ 0 & \phi_1 \end{pmatrix} \tag{30}$$

or more generally, for a rank N block, i.e. $J^N = 0$, $J^{N-1} \neq 0$

$$v \mapsto \sum_{i=0}^{N-1} \phi_i(z) J^{N-1-i} =: \boldsymbol{\phi}(z, J) \tag{31}$$

The l_n of (17) read exactly the same, except now act by matrix multiplication, rather than by multiplication on a vector.

This notation will be useful for considering two-point functions. Let $\boldsymbol{\phi}(z, J)$, $\boldsymbol{\psi}(w, K)$ where $J^M = 0$, $J^{M-1} \neq 0$ and $K^N = 0$, $K^{N-1} \neq 0$ be two logarithmic primaries. Then the two point function reads

$$\boldsymbol{f}(z, w, J, K) = \langle 0 | \boldsymbol{\phi}(z, J) \otimes \boldsymbol{\psi}(w, K) 0 \rangle \tag{32}$$

where the tensor product $\otimes$ is between the vector space of $M \times M$ matrices and $N \times N$ matrices.

One can ask what conditions the symmetry generators $l_0, l_{\pm 1}$ impose on f. To this end, it is useful to work in co-ordinates $x = z - w$, $y = x + w$. The l_{-1} symmetry then imposes

$$(l_{-1}^{(1)} + l_{-1}^{(2)}) \boldsymbol{f} = 2 \frac{\partial}{\partial y} \boldsymbol{f} = 0 \tag{33}$$

yielding $\boldsymbol{f} = \boldsymbol{f}(x, J, K)$. The remaining conditions then read

$$(l_0^{(1)} + l_0^{(2)}) \boldsymbol{f} = \Big(\mathbb{I} \otimes \mathbb{I} (x \tfrac{\partial}{\partial x} + h_1 + h_2) + J \otimes \mathbb{I} + \mathbb{I} \otimes K \Big) \boldsymbol{f} = 0 \tag{34}$$

$$(l_1^{(1)} + l_1^{(2)}) \boldsymbol{f} = \Big(y (l_0^{(1)} + l_0^{(2)}) + x (\mathbb{I} \otimes \mathbb{I} (h_1 - h_2) + J \otimes \mathbb{I} - \mathbb{I} \otimes K) \Big) \boldsymbol{f} = 0 \tag{35}$$

Since $\boldsymbol{f}$ is a function of J and K, it can be expanded out into a 'polynomial' in $J^m \otimes K^n$. (34) then reads as MN coupled first order ordinary differential equations in x, and hence should give MN independent solutions. (34) can be rewritten, using (35) as

$$\Big(\mathbb{I} \otimes \mathbb{I} (x \tfrac{\partial}{\partial x} + 2h_1) + 2J \otimes \mathbb{I} \Big) \boldsymbol{f} = 0 \tag{36}$$

which has solution

$$\boldsymbol{f} = \boldsymbol{C}(J, K) x^{-2(\mathbb{I} \otimes \mathbb{I} h_1 + J \otimes \mathbb{I})} = \boldsymbol{C}(J, K) x^{-2h_1} \sum_{k=0}^{M-1} \frac{1}{k!} J^k \otimes \mathbb{I} (-2 \log x)^k \tag{37}$$

Here, $\boldsymbol{C}$ can be expanded as $\boldsymbol{C} = \sum_{m,n=0}^{M-1,N-1} C_{m,n} J^m \otimes K^n$, and hence yields MN independent parameters, as required for the general solution. Using the solution given by (37), the condition given by (35) then reduces to

$$(h_1 - h_2)C_{m,n} + C_{(m-1),n} - C_{m,(n-1)} = 0 \tag{38}$$

for each m, n where $C_{-1,n} = C_{m,-1} = 0$. (38) only yields non-trivial solutions for $h_1 - h_2 = 0$. Now, M is not necessarily equal to N, and without loss of generality, one can choose $M \leq N$. (38) then yields $C_{i,j} = 0$ for $i + j < N - 1$. Hence, there are only M free parameters in $\boldsymbol{C}$, which are given by $C_{m,N-1}$.

Plugging in values for M and N can give rise to familiar solutions. Consider $M = N = 2$, and set $C_{1,0} = a$ and $C_{1,1} = b$. Using a shorthand of suppressing the $\mathbb{I}$ and $\otimes$ symbols, one has[9], $\boldsymbol{C} = (J + K)a + JKb$. Hence

$$\boldsymbol{C}x^{-2(h_1+J)} = x^{-2h_1}\Big((J + K)a + JK(b - 2a\log x)\Big) \tag{39}$$

Other values of M and N yield less familiar solutions. For an example of different Jordan block sizes, consider $M = 2$, $N = 3$. One has $\boldsymbol{C} = (K^2 + JK)a + JK^2b$, yielding

$$\boldsymbol{C}x^{-2(h_1+J)} = x^{-2h_1}\Big((K^2 + JK)a + JK^2(b - 2a\log x)\Big) \tag{40}$$

To the author's knowledge, this is the first time a two-point function has been found where the Jordan blocks differ in size.

5 $N = 1$ Logarithmic Conformal Field Theory

The same game can be played in the $N = 1$ case. A conformal condition in two dimensional Superconformal Field Theory is normally specified by a one-form being preserved up to an overall scale factor. Usually, the preserved one-form is of the form $\omega = dz - \sum_i d\theta_i\theta_i$. The conformal condition then reads $f : (z, \theta_i) \mapsto (z', \theta_i')$ is conformal if $f^*\omega = \omega\kappa$ for some function $\kappa = \kappa(z, \theta_i)$, and f is invertible. The invertibility condition implies that κ has body, and that $\frac{1}{\kappa}$ is well defined. For the $N = 1$, a conformal transformation is given by a transformation that preserves $\omega = dz - d\theta\theta =: dz + \theta d\theta$ up to an overall scale factor $\kappa(z, \theta)$. Primary fields can be defined [10] as sections of the locally rank 1 sheaf given by ω^h. Just as before, one can instead consider ω^{h+J}. Going through the same machinery, this gives the transformation laws for $J^2 = 0$, and a conformal transformation f,

$$\begin{aligned} \Phi_0'(z,\theta) &= (D\theta')^{2h}\Big((\Phi_0 \circ f)(z,\theta) + 2\log(D\theta')(\Phi_1 \circ f)(z,\theta)\Big) \\ \Phi_1'(z,\theta) &= (D\theta')^{2h}(\Phi_1 \circ f)(z,\theta) \end{aligned} \tag{41}$$

which gives rise to infinitesimal transformations, with $n \in \mathbb{Z}, r \in \mathbb{Z} + \frac{1}{2}$

$$[L_n, \Phi_0] = h(n+1)z^n\Phi_0 + z^{n+1}\partial\Phi_0 + \frac{n+1}{2}z^n\theta\partial_\theta\Phi_0 + (n+1)z^n\Phi_1$$

$$
\begin{aligned}
[G_r, \Phi_0] &= h(2r+1)z^{r-\frac{1}{2}}\theta\Phi_0 + z^{r+\frac{1}{2}}\theta\partial\Phi_0 - z^{r+\frac{1}{2}}\partial_\theta\Phi_0 + (2r+1)z^{r-\frac{1}{2}}\theta\Phi_1 \\
[L_n, \Phi_1] &= h(n+1)z^n\Phi_1 + z^{n+1}\partial\Phi_1 + \frac{n+1}{2}z^n\theta\partial_\theta\Phi_1 \\
[G_r, \Phi_1] &= h(2r+1)z^{r-\frac{1}{2}}\theta\Phi_1 + z^{r+\frac{1}{2}}\theta\partial\Phi_1 - z^{r+\frac{1}{2}}\partial_\theta\Phi_1
\end{aligned} \tag{42}
$$

which are the well known commutators for a logarithmic $N = 1$ Neveu-Schwarz theory with a rank two block. These were first found in [6] by a different method, namely consistency with the Jacobi identity, and requiring L_{-1} to generate translations. Acting on the vacuum, and letting $(z, \theta) \to (0, 0)$ gives

$$
L_0|\Phi_0\rangle = h|\Phi_0\rangle + |\Phi_1\rangle, \qquad L_0|\Phi_1\rangle = h|\Phi_1\rangle, \qquad L_n|\Phi_i\rangle = 0, \qquad G_r|\Phi_i\rangle = 0 \tag{43}
$$

for $i = 1, 2$, and $n, r > 0$.

To study the Ramond case, one might consider $\omega = dz + z\theta d\theta$. This leads to

$$
\begin{aligned}
\Phi_0'(z,\theta) &= \left(\frac{z'}{z}(D\theta')^2\right)^h \left((\Phi_0 \circ f)(z,\theta) + \log\left(\frac{z'}{z}(D\theta')^2\right)(\Phi_1 \circ f)(z,\theta)\right) \\
\Phi_1'(z,\theta) &= \left(\frac{z'}{z}(D\theta')^2\right)^h (\Phi_1 \circ f)(z,\theta)
\end{aligned} \tag{44}
$$

and, with $n, r \in \mathbb{Z}$,

$$
\begin{aligned}
[L_n, \Phi_0] &= h(n+1)z^n\Phi_0 + z^{n+1}\partial\Phi_0 + \frac{n}{2}z^n\theta\partial_\theta\Phi_0 + (n+1)z^n\Phi_1 \\
[G_r, \Phi_0] &= h(2r+1)z^r\theta\Phi_0 + z^{r+1}\theta\partial\Phi_0 - z^r\partial_\theta\Phi_0 + (2r+1)z^r\theta\Phi_1 \\
[L_n, \Phi_1] &= h(n+1)z^n\Phi_1 + z^{n+1}\partial\Phi_1 + \frac{n}{2}z^n\theta\partial_\theta\Phi_1 \\
[G_r, \Phi_1] &= h(2r+1)z^r\theta\Phi_1 + z^{r+1}\theta\partial\Phi_1 - z^r\partial_\theta\Phi_1
\end{aligned} \tag{45}
$$

By acting on the vacuum and looking at $(z, \theta) \to (0, 0)$, the formulae

$$
L_0|\Phi_0\rangle = h|\Phi_0\rangle + |\Phi_1\rangle, \qquad L_0|\Phi_1\rangle = h|\Phi_1\rangle, \qquad L_n|\Phi_i\rangle = 0, \qquad G_r|\Phi_i\rangle = 0 \tag{46}
$$

for $i = 1, 2$, and $n, r > 0$ are still obtained. Now,

$$
\lim_{z,\theta\to 0}[G_0, \Phi_0]|0\rangle = \lim_{z,\theta\to 0}\Big(h\theta\Phi_0 + z\theta\partial\Phi_0 - \partial_\theta\Phi_0 + \theta\Phi_1\Big)|0\rangle = -|\partial_\theta\Phi_0\rangle \tag{47}
$$

and hence the usual G_0 action on the highest weight in a Ramond theory does not appear to be affected.

6 $N = 2$ Conformal Field Theory

For the $N = 2$ case, the preserved one-form is $\omega = dz - \sum_{i=1}^{2} d\theta_i\theta_i$. The conformal condition reads $f : (z, \theta_i) \mapsto (z', \theta_i')$ is conformal if $f^*\omega = \omega\kappa$ for some function κ, and f is invertible. From this, it can be deduced that the superderivatives $D_i = \frac{\partial}{\partial\theta_i} + \theta_i\frac{\partial}{\partial z}$ enjoy the property $\sum_j (D_i\theta_j')(D_k\theta_j') = \delta_{ik}\kappa$ and hence $\frac{D_i\theta_j'}{\sqrt{\kappa}}$ is an even Grassmann complex orthogonal matrix.

In an $N = 2$ Neveu-Schwarz theory, where $i = 1, 2$ in ω, the infinitesimal transformations can be represented by the differential operators

$$l_m = -z^m\left(z\frac{\partial}{\partial z} + \frac{1}{2}(m+1)\theta_i\frac{\partial}{\partial\theta_i}\right) \qquad t_m = z^m\left(\theta_1\frac{\partial}{\partial\theta_2} - \theta_2\frac{\partial}{\partial\theta_1}\right)$$
$$g_r^i = z^{r-\frac{1}{2}}\left(z\theta_i\frac{\partial}{\partial z} - z\frac{\partial}{\partial\theta_i} + (r+\tfrac{1}{2})\theta_i\theta_j\frac{\partial}{\partial\theta_j}\right) \tag{48}$$

The t_m term represents a $O(2)$ symmetry on the space of functions. This term can be diagonalized by the change of co-ordinates $m : (\theta_1, \theta_2) \mapsto (\theta^+, \theta^-)$, given by

$$\theta^+ = \frac{1}{\sqrt{2}}(\theta_1 + i\theta_2) \qquad \theta^- = \frac{1}{\sqrt{2}}(\theta_1 - i\theta_2) \tag{49}$$

This is only a change of co-ordinates, and not a conformal transformation. The co-ordinate change amounts to studying those transformations that preserve the one-form $\nu = dz - d\theta^+\theta^- - d\theta^-\theta^+$. The condition for a transformation $g : (z, \theta^+, \theta^-) \mapsto (z', \theta^{+\prime}, \theta^{-\prime})$ to be conformal is

$$g^*\nu = \nu\kappa \tag{50}$$

with g invertible. Consider now a conformal (wrt ν) transformation $g : (z, \theta^+, \theta^-) \mapsto (z', \theta^{+\prime}, \theta^{-\prime})$ with conformal scaling factor κ. Then $m^{-1}g^*m$ gives a conformal transformation wrt ω, with conformal scaling factor $\kappa \circ m$.

$$\begin{aligned} m^{-1}g^*m(\omega(z', \theta_1', \theta_2')) = m^{-1}g^*(\nu(z', \theta^{+\prime}, \theta^{-\prime})) &= m^{-1}(\nu(z, \theta^+, \theta^-)\kappa(z, \theta^+, \theta^-)) \\ &= \omega(z, \theta_1, \theta_2)(\kappa \circ m)(z, \theta_1, \theta_2) \end{aligned} \tag{51}$$

A similar calculation can be done starting with a conformal transformation wrt ω, hence the groups of transformations for the two conformal conditions are isomorphic. The conformal condition implies that

$$\kappa = \partial z' + \theta^{+\prime}\partial\theta^{-\prime} + \theta^{-\prime}\partial\theta^{+\prime} \tag{52}$$
$$D_+z' = \theta^{+\prime}D_+\theta^{-\prime} + \theta^{-\prime}D_+\theta^{+\prime} \tag{53}$$
$$D_-z' = \theta^{+\prime}D_-\theta^{-\prime} + \theta^{-\prime}D_-\theta^{+\prime} \tag{54}$$

where $D_\pm = \frac{\partial}{\partial\theta^\pm} + \theta^\mp\frac{\partial}{\partial z}$. From this definition of $D_\pm$, the graded commutators $[D_+, D_+] = 0$, $[D_-, D_-] = 0$ and $[D_+, D_-] = \frac{\partial}{\partial z}$ can be calculated, from which it can be seen that

$$(D_+\theta^{+\prime})(D_-\theta^{-\prime}) + (D_+\theta^{-\prime})(D_-\theta^{+\prime}) = \kappa \tag{55}$$
$$(D_+\theta^{+\prime})(D_+\theta^{-\prime}) = 0 \tag{56}$$
$$(D_-\theta^{-\prime})(D_-\theta^{+\prime}) = 0 \tag{57}$$

Under the conformal transformation, the superderivatives transform as

$$\begin{pmatrix} D_+ \\ D_- \end{pmatrix} = \begin{pmatrix} D_+\theta^{+\prime} & D_+\theta^{-\prime} \\ D_-\theta^{+\prime} & D_-\theta^{-\prime} \end{pmatrix} \begin{pmatrix} D'_+ \\ D'_- \end{pmatrix} = M \begin{pmatrix} D'_+ \\ D'_- \end{pmatrix} \tag{58}$$

Consider now the product of matrices

$$\begin{pmatrix} D_+\theta^{+\prime} & D_+\theta^{-\prime} \\ D_-\theta^{+\prime} & D_-\theta^{-\prime} \end{pmatrix} \begin{pmatrix} D_-\theta^{-\prime} & D_+\theta^{-\prime} \\ D_-\theta^{+\prime} & D_+\theta^{+\prime} \end{pmatrix} = \kappa \mathbb{I}_2 \tag{59}$$

Taking determinants, it can be seen that $\det M = \pm\kappa$. In the $\det M = +\kappa$ case, $\kappa = (D_+\theta^{+\prime})(D_-\theta^{-\prime})$. Therefore, $D_+\theta^{+\prime}$ and $D_-\theta^{-\prime}$ must both have body, implying $D_-\theta^{+\prime} = 0 = D_+\theta^{-\prime}$. In the $\det M = -\kappa$ case, $\kappa = (D_+\theta^{-\prime})(D_-\theta^{+\prime})$, and similarly to the previous case, $D_+\theta^{+\prime} = 0 = D_-\theta^{-\prime}$. These two cases can be related to the $O(2)$ symmetry. Explicitly, the first case has

$$\frac{M}{\sqrt{\kappa}} = \begin{pmatrix} \left(\frac{D_+\theta^{+\prime}}{D_-\theta^{-\prime}}\right)^{\frac{1}{2}} & 0 \\ 0 & \left(\frac{D_+\theta^{+\prime}}{D_-\theta^{-\prime}}\right)^{-\frac{1}{2}} \end{pmatrix} \tag{60}$$

This gives rise to a decomposable representation of $SO(2)$. The second case gives

$$\frac{M}{\sqrt{\kappa}} = \begin{pmatrix} 0 & \left(\frac{D_+\theta^{-\prime}}{D_-\theta^{+\prime}}\right)^{\frac{1}{2}} \\ \left(\frac{D_+\theta^{-\prime}}{D_-\theta^{+\prime}}\right)^{-\frac{1}{2}} & 0 \end{pmatrix} \tag{61}$$

which is in a region of $O(2)$ disconnected from the identity. If one is only concerned with those transformations connected to the identity, only the first case is of concern. Looking at only the transformations connected to the identity is sufficient to study the related conformal algebra. To this end, as far as the $SO(2)$ symmetry is concerned, only the transformation rule of $\nu^{-\frac{1}{2}} \otimes D_+ =: \delta$ is needed. For transformations only in the connected part of the conformal group, this locally gives a rank 1 sheaf over a graded Riemann sphere, and hence the restriction maps give rise to an abelian group. Looking at the representations of this group, primary superfields can be constructed as sections $\mathbf{\Phi}$ of $\nu^h \otimes \delta^q$ which, under pull-back, then yields the familiar transformation law

$$\begin{aligned} (f^*\mathbf{\Phi})(z,\theta^+,\theta^-) &= \kappa^h \left(\frac{D_+\theta^{+\prime}}{D_-\theta^{-\prime}}\right)^{\frac{q}{2}} (\Phi\circ f)(z,\theta^+,\theta^-)\nu^h \otimes \delta^q \\ &= (D_+\theta^{+\prime})^{h+\frac{q}{2}}(D_-\theta^{-\prime})^{h-\frac{q}{2}}\Phi(z',\theta^{+\prime},\theta^{-\prime})\nu^h\otimes\delta^q \qquad (62) \\ &=: \Phi'(z,\theta^+,\theta^-)\nu^h\otimes\delta^q \qquad (63) \end{aligned}$$

7 $N = 2$ Logarithmic Conformal Field Theory

Recall for the bosonic case, a logarithmic CFT was found by formally replacing h with $h + J$ in the exponent of dz. In the $N = 2$ case there are two exponents in $\nu^h \otimes \delta^q$, h and q, which be replaced by $h\mathbb{I}_{V_A} + A$ and $q\mathbb{I}_{V_B} + B$ respectively, where A and B are nilpotent matrices on finite dimensional vector spaces V_A and V_B respectively. Then a section, $\mathbf{\Phi}$, of

$$\nu^{h+A} \otimes \delta^{q+B} \tag{64}$$

will be $V_A \otimes V_B$ valued. Under pull-back, one obtains

$$\begin{aligned} f^*\mathbf{\Phi} &= f^*(\nu^{h\mathbb{I}_{V_A}+A} \otimes \delta^{q\mathbb{I}_{V_B}+B}\Phi) \\ &= \left(\nu^{h\mathbb{I}_{V_A}+A}((D_+\theta^+)(D_-\theta^-))^{h\mathbb{I}_{V_A}+A} \otimes \delta^{q\mathbb{I}_{V_B}+B}\left(\frac{D_+\theta^+}{D_-\theta^-}\right)^{\frac{q\mathbb{I}_{V_B}+B}{2}}\right)(\Phi\circ f) \\ &= \left(\nu^{h\mathbb{I}_{V_A}+A} \otimes \delta^{q\mathbb{I}_{V_B}+B}\right)\left(((D_+\theta^+)(D_-\theta^-))^{h\mathbb{I}_{V_A}+A} \otimes \left(\frac{D_+\theta^+}{D_-\theta^-}\right)^{\frac{q\mathbb{I}_{V_B}+B}{2}}\right)(\Phi\circ f) \end{aligned} \tag{65}$$

This yields the transformation law

$$\Phi'(z,\theta^+,\theta^-) = \left(((D_+\theta^+)(D_-\theta^-))^{h\mathbb{I}_{V_A}+A} \otimes \left(\frac{D_+\theta^+}{D_-\theta^-}\right)^{\frac{q\mathbb{I}_{V_B}+B}{2}}\right)\Phi(z',\theta^{+\prime},\theta^{-\prime}) \tag{66}$$

Using the infinitesimal transformations given by the superconformal condition, (66) gives the vector fields

$$\begin{aligned} l_m\Phi &= -z^m\Bigg(\left(z\partial + \frac{1}{2}(m+1)(\theta^+\partial_+ + \theta^-\partial_-) + h(m+1) - \frac{q}{2}m(m+1)\frac{\theta^+\theta^-}{z}\right)\mathbb{I}\otimes\mathbb{I} \\ &\qquad +(m+1)(A\otimes\mathbb{I}) - m(m+1)\frac{\theta^+\theta^-}{2z}(\mathbb{I}\otimes B)\Bigg)\Phi \\ g_{+r}\Phi &= z^{r-\frac{1}{2}}\Bigg(\left(z\theta^-\partial - z\partial_+ + (r+\frac{1}{2})\theta^-\theta^+\partial_+ + (2h+q)(r+\frac{1}{2})\theta^-\right)\mathbb{I}\otimes\mathbb{I} \\ &\qquad +\theta^-(r+\frac{1}{2})(2A\otimes\mathbb{I} + \mathbb{I}\otimes B)\Bigg)\Phi \\ g_{-r}\Phi &= z^{r-\frac{1}{2}}\Bigg(\left(z\theta^+\partial - z\partial_- + (r+\frac{1}{2})\theta^+\theta^-\partial_- + (2h-q)(r+\frac{1}{2})\theta^+\right)\mathbb{I}\otimes\mathbb{I} \\ &\qquad +\theta^+(r+\frac{1}{2})(2A\otimes\mathbb{I} - \mathbb{I}\otimes B)\Bigg)\Phi \\ j_m\Phi &= -z^m\Bigg(\left(\theta^+\partial_+ - \theta^-\partial_- - 2mh\frac{\theta^+\theta^-}{z} + q\right)\mathbb{I}\otimes\mathbb{I} - 2m\frac{\theta^+\theta^-}{z}(A\otimes\mathbb{I}) + (\mathbb{I}\otimes B)\Bigg)\Phi \end{aligned} \tag{67}$$

These vector fields then obey the graded commutation relations of the centreless $N=2$ algebra

$$[l_m, l_n] = (m-n)l_{m+n} \qquad [l_m, g_{\pm r}] = (\tfrac{m}{2} - r)g_{\pm(m+r)} \qquad [l_m, t_n] = -nj_{m+n}$$

$$[g_{+r}, g_{-s}] = 2l_{r+s} + (r-s)j_{r+s} \qquad [g_{\pm r}, g_{\pm s}] = 0$$

$$[j_m, g_{\pm r}] = \pm g_{\pm(m+r)} \qquad [j_m, j_n] = 0 \tag{68}$$

8 Two Point Function

Consider first the case $A = 0 = B$. Then the symmetry generators can be used to calculate

$$f(Z_1, Z_2) = f(z, \theta_1^+, \theta_1^-, w, \theta_2^+, \theta_2^-) = \langle \Phi_1(Z_1)\Phi_2(Z_2)\rangle. \tag{69}$$

Consider the change of variables

$$\begin{array}{cc} Z_{12} = (z - w) - (\theta_1^+\theta_2^- + \theta_1^-\theta_2^+), & W_{12} = (z + w) - (\theta_1^+\theta_2^- + \theta_1^-\theta_2^+) \\ \theta_{12}^+ = \theta_1^+ - \theta_2^+, \quad \theta_{12}^- = \theta_1^- - \theta_2^-, & \xi_{12}^+ = \theta_1^+ + \theta_2^+, \quad \xi_{12}^- = \theta_1^- + \theta_2^- \end{array} \tag{70}$$

Then

$$\begin{array}{cc} \frac{\partial}{\partial Z_{12}} = \frac{1}{2}\Big(\frac{\partial}{\partial z} - \frac{\partial}{\partial w}\Big), & \frac{\partial}{\partial W_{12}} = \frac{1}{2}\Big(\frac{\partial}{\partial z} + \frac{\partial}{\partial w}\Big) \\ \frac{\partial}{\partial \theta_{12}^+} = \frac{1}{2}\Big(\frac{\partial}{\partial \theta_1^+} - \frac{\partial}{\partial \theta_2^+} + (\theta_2^- + \theta_1^-)\frac{\partial}{\partial z}\Big), & \frac{\partial}{\partial \theta_{12}^-} = \frac{1}{2}\Big(\frac{\partial}{\partial \theta_1^-} - \frac{\partial}{\partial \theta_2^-} + (\theta_2^+ + \theta_1^+)\frac{\partial}{\partial z}\Big) \\ \frac{\partial}{\partial \xi_{12}^+} = \frac{1}{2}\Big(\frac{\partial}{\partial \theta_1^+} + \frac{\partial}{\partial \theta_2^+} + (\theta_2^- - \theta_1^-)\frac{\partial}{\partial z}\Big), & \frac{\partial}{\partial \xi_{12}^-} = \frac{1}{2}\Big(\frac{\partial}{\partial \theta_1^-} + \frac{\partial}{\partial \theta_2^-} + (\theta_2^+ - \theta_1^+)\frac{\partial}{\partial z}\Big) \end{array} \tag{71}$$

Now consider the action of the lie algebra on f.

$$\begin{aligned} (l_{-1}^{(1)} + l_{-1}^{(2)})f = 0 &= \Big(\frac{\partial}{\partial z} + \frac{\partial}{\partial w}\Big)f = 2\frac{\partial}{\partial W_{12}}f \\ (g_{+,-\frac{1}{2}}^{(1)} + g_{+,-\frac{1}{2}}^{(2)})f = 0 &= \Big(\frac{\partial}{\partial \theta_1^+} - \theta_1^-\frac{\partial}{\partial z} + \frac{\partial}{\partial \theta_2^+} - \theta_2^-\frac{\partial}{\partial w}\Big)f = 2\frac{\partial}{\partial \xi_{12}^+}f \\ (g_{-,-\frac{1}{2}}^{(1)} + g_{-,-\frac{1}{2}}^{(2)})f = 0 &= \Big(\frac{\partial}{\partial \theta_1^-} - \theta_1^+\frac{\partial}{\partial z} + \frac{\partial}{\partial \theta_2^-} - \theta_2^+\frac{\partial}{\partial w}\Big)f = 2\frac{\partial}{\partial \xi_{12}^-}f \end{aligned} \tag{72}$$

Hence $f = f(\theta_{12}^+, \theta_{12}^-, Z_{12})$. Similarly

$$(l_0^{(1)} + l_0^{(2)})f = 0 = \tag{73}$$
$$\Big(h_1 + h_2 + Z_{12}\frac{\partial}{\partial Z_{12}} + W_{12}\frac{\partial}{\partial W_{12}} + \frac{1}{2}\Big(\theta_{12}^+\frac{\partial}{\partial \theta_{12}^+} + \xi_{12}^+\frac{\partial}{\partial \xi_{12}^+} + \theta_{12}^-\frac{\partial}{\partial \theta_{12}^-} + \xi_{12}^-\frac{\partial}{\partial \xi_{12}^-}\Big)\Big)f$$

yielding

$$f = a_0 Z_{12}^{-h_1-h_2} + a_+\theta_{12}^+ Z_{12}^{-h_1-h_2-\frac{1}{2}} + a_-\theta_{12}^- Z_{12}^{-h_1-h_2-\frac{1}{2}} + a_{+-}\theta_{12}^+\theta_{12}^- Z_{12}^{-h_1-h_2-1} \tag{74}$$

where a_0, a_{+-} are graded even constants, and $a_\pm$ are graded odd constants. Applying the condition

$$(j_0^{(1)} + j_0^{(2)})f = 0 = \Big(q_1 + q_2 + \theta_{12}^+\frac{\partial}{\partial \theta_{12}^+} + \xi_{12}^+\frac{\partial}{\partial \xi_{12}^+} - \theta_{12}^-\frac{\partial}{\partial \theta_{12}^-} - \xi_{12}^-\frac{\partial}{\partial \xi_{12}^-}\Big)f \tag{75}$$

yields the solutions either $(q_1 + q_2) = 0 = a_\pm$, or $(q_1 + q_2 + 1) = 0 = a_0 = a_{+-} = a_-$, or $(q_1 + q_2 - 1) = 0 = a_0 = a_{+-} = a_+$. Since the commutator of l_1 with $g_{\pm,-\frac{1}{2}}$ gives $g_{\pm,\frac{1}{2}}$,

only the l_1 condition need be applied.

$$
\begin{aligned}
&(l_1^{(1)} + l_1^{(2)})f = 0 = \\
&\frac{1}{4}\Big[(W_{12}^2 + Z_{12}^2)\frac{\partial}{\partial W_{12}} + 4W_{12}Z_{12}\frac{\partial}{\partial Z_{12}} + 2Z_{12}(\theta_{12}^+\xi_{12}^- + \theta_{12}^-\xi_{12}^+)\Big(\frac{\partial}{\partial Z_{12}} + \frac{\partial}{\partial W_{12}}\Big) \\
&-\xi_{12}^+(\xi_{12}^- + \theta_{12}^-)\theta_{12}^+\Big(\frac{\partial}{\partial \xi_{12}^+} + \frac{\partial}{\partial \theta_{12}^+}\Big) - \xi_{12}^-(\xi_{12}^+ + \theta_{12}^+)\theta_{12}^-\Big(\frac{\partial}{\partial \xi_{12}^-} + \frac{\partial}{\partial \theta_{12}^-}\Big) \\
&2W_{12}(\xi_{12}^+\frac{\partial}{\partial \xi_{12}^+} + \theta_{12}^+\frac{\partial}{\partial \theta_{12}^+} + \xi_{12}^-\frac{\partial}{\partial \xi_{12}^-} + \theta_{12}^-\frac{\partial}{\partial \theta_{12}^-}) + \\
&2Z_{12}(\xi_{12}^+\frac{\partial}{\partial \theta_{12}^+} + \theta_{12}^+\frac{\partial}{\partial \xi_{12}^+} + \xi_{12}^-\frac{\partial}{\partial \theta_{12}^-} + \theta_{12}^-\frac{\partial}{\partial \xi_{12}^-}) + 4W_{12}(h_1 + h_2) + 4Z_{12}(h_1 - h_2) + \\
&4h_1(\theta_{12}^+\xi_{12}^- + \theta_{12}^-\xi_{12}^+) - (q_1 + q_2)(\xi_{12}^+\xi_{12}^- + \theta_{12}^+\theta_{12}^-) - (q_1 - q_2)(\theta_{12}^+\xi_{12}^- + \xi_{12}^+\theta_{12}^-)\Big]f = \\
&\frac{1}{4}\Big[4W_{12}(l_0^{(1)} + l_0^{(2)}) + 2Z_{12}(\theta_{12}^+\xi_{12}^- + \theta_{12}^-\xi_{12}^+)\Big(\frac{\partial}{\partial Z_{12}}\Big) - \xi_{12}^+(\xi_{12}^- + \theta_{12}^-)\theta_{12}^+\Big(\frac{\partial}{\partial \theta_{12}^+}\Big) - \\
&\xi_{12}^-(\xi_{12}^+ + \theta_{12}^+)\theta_{12}^-\Big(\frac{\partial}{\partial \theta_{12}^-}\Big) + 2Z_{12}(\xi_{12}^+\frac{\partial}{\partial \theta_{12}^+} + \xi_{12}^-\frac{\partial}{\partial \theta_{12}^-}) + 4Z_{12}(h_1 - h_2) + \\
&4h_1(\theta_{12}^+\xi_{12}^- + \theta_{12}^-\xi_{12}^+) - (q_1 + q_2)(\xi_{12}^+\xi_{12}^- + \theta_{12}^+\theta_{12}^-) - (q_1 - q_2)(\theta_{12}^+\xi_{12}^- + \xi_{12}^+\theta_{12}^-)\Big]f \\
= \quad &\frac{1}{4}\Big[2(2W_{12} + \theta_{12}^+\xi_{12}^- + \theta_{12}^-\xi_{12}^+)(l_0^{(1)} + l_0^{(2)}) - \\
&(\xi_{12}^+\xi_{12}^- + \theta_{12}^+\theta_{12}^- + \theta_{12}^+\xi_{12}^- + \xi_{12}^+\theta_{12}^-)(j_0^{(1)} + j_0^{(2)}) + 2(h_1 - h_2)(2Z_{12} + \theta_{12}^+\xi_{12}^- + \\
&\theta_{12}^-\xi_{12}^+) + 2Z_{12}(\xi_{12}^+\frac{\partial}{\partial \theta_{12}^+} + \xi_{12}^-\frac{\partial}{\partial \theta_{12}^-}) - (\theta_{12}^+\xi_{12}^- + \theta_{12}^-\xi_{12}^+)(\theta_{12}^+\frac{\partial}{\partial \theta_{12}^+} + \theta_{12}^-\frac{\partial}{\partial \theta_{12}^-}) + \\
&2q_2(\theta_{12}^+\xi_{12}^- + \xi_{12}^+\theta_{12}^-)\Big]f
\end{aligned} \tag{76}
$$

which then yields only one possibly non-trivial solution, namely $h_1 - h_2 = q_1 + q_2 = 0 = a_\pm$ and $a_{+-} = -q_2 a_0$.

Consider now non-zero A, B, which amounts to replacing h_1 by $h_1 + J$, h_2 by $h_2 + K$, q_1 by $q_1 + P$ and q_2 by $q_2 + Q$. The extra parameters have the properties that $J^M, K^N, P^R, Q^S = 0$ and $J^{M-1}, K^{N-1}, P^{R-1}, Q^{S-1} \neq 0$. So as not to overly clutter the notation, tensor product signs will be omitted. (72) remains unchanged, and hence $\boldsymbol{f} = \boldsymbol{f}(\theta_{12}^+, \theta_{12}^-, Z_{12}, J, K, P, Q)$. (73) is modified to

$$
\Big(h_1 + J + h_2 + K + Z_{12}\frac{\partial}{\partial Z_{12}} + \frac{1}{2}\Big(\theta_{12}^+\frac{\partial}{\partial \theta_{12}^+} + \theta_{12}^-\frac{\partial}{\partial \theta_{12}^-}\Big)\Big)\boldsymbol{f} = 0 \tag{77}
$$

which, similarly to the bosonic case, has solution

$$
\boldsymbol{f} = \boldsymbol{a}_0 Z_{12}^{-\Delta} + \boldsymbol{a}_+\theta_{12}^+ Z_{12}^{-\Delta-\frac{1}{2}} + \boldsymbol{a}_-\theta_{12}^- Z_{12}^{-\Delta-\frac{1}{2}} + \boldsymbol{a}_{+-}\theta_{12}^+\theta_{12}^- Z_{12}^{-\Delta-1} \tag{78}
$$

where $\Delta = h_1 + J + h_2 + K$, and the prefactors have dependence $\boldsymbol{a} = \boldsymbol{a}(J, K, P, Q)$. (75) becomes

$$
\Big(q_1 + P + q_2 + Q + \theta_{12}^+\frac{\partial}{\partial \theta_{12}^+} - \theta_{12}^-\frac{\partial}{\partial \theta_{12}^-}\Big)\boldsymbol{f} = 0 \tag{79}
$$

This yields the conditions on the prefactors

$$
\begin{aligned}
&(q_1 + q_2 + P + Q)\boldsymbol{a}_0 = 0, \qquad (q_1 + q_2 + P + Q + 1)\boldsymbol{a}_+ = 0, \\
&(q_1 + q_2 + P + Q - 1)\boldsymbol{a}_- = 0, \qquad (q_1 + q_2 + P + Q)\boldsymbol{a}_{+-} = 0
\end{aligned} \tag{80}
$$

Now, since $(P + Q)$ is nilpotent, if $q_1 + q_2 \neq 0, \pm 1$, the above relations can be inverted to show that all the $\boldsymbol{a} = 0$. The possibly non-trivial solutions are given by

$$
q_1 + q_2 = 0 \;\Rightarrow\; \boldsymbol{a}_\pm = 0, \quad (P+Q)\boldsymbol{a}_0 = 0, \quad (P+Q)\boldsymbol{a}_{+-} = 0 \tag{81}
$$

$$
q_1 + q_2 = 1 \;\Rightarrow\; \boldsymbol{a}_0 = \boldsymbol{a}_+ = \boldsymbol{a}_{+-} = 0, \quad (P+Q)\boldsymbol{a}_- = 0 \tag{82}
$$

$$
q_1 + q_2 = -1 \;\Rightarrow\; \boldsymbol{a}_0 = \boldsymbol{a}_- = \boldsymbol{a}_{+-} = 0, \quad (P+Q)\boldsymbol{a}_+ = 0 \tag{83}
$$

(76) now reads

$$
\begin{aligned}
&\Big(2(h_1 - h_2 + J - K)(2Z_{12} + \theta^+_{12}\xi^-_{12} + \theta^-_{12}\xi^+_{12}) + 2(q_2 + Q)(\theta^+_{12}\xi^-_{12} + \xi^+_{12}\theta^-_{12}) + \\
&\quad 2Z_{12}\Big(\xi^+_{12}\frac{\partial}{\partial\theta^+_{12}} + \xi^-_{12}\frac{\partial}{\partial\theta^-_{12}}\Big) - (\theta^+_{12}\xi^-_{12} + \theta^-_{12}\xi^+_{12})\Big(\theta^+_{12}\frac{\partial}{\partial\theta^+_{12}} + \theta^-_{12}\frac{\partial}{\partial\theta^-_{12}}\Big)\Big)\boldsymbol{f} = 0
\end{aligned} \tag{84}
$$

For $q_1 + q_2 = \pm 1$, this yields only the trivial solution, and for $q_1 + q_2 = 0$ yields $(q_2 + Q)\boldsymbol{a}_0 + \boldsymbol{a}_{+-} = 0 = h_1 - h_2$ and $(J - K)\boldsymbol{a}_0 = 0$. Hence

$$
\boldsymbol{f} = \boldsymbol{a}_0\Big(Z_{12}^{-\Delta} - (q_2 + Q)\theta^+_{12}\theta^-_{12}Z_{12}^{-\Delta-1}\Big) \tag{85}
$$

subject to $(J - K)\boldsymbol{a}_0 = 0 = (P + Q)\boldsymbol{a}_0$. $\boldsymbol{a}_0$ thus has $\min(M, N) \times \min(R, S)$ free parameters. In particular, allowing j_0 to be non-diagonalizable does not seem to introduce any extra logarithms into the two point function. This is perhaps not too surprising, since in the two point function, Q only appears in conjunction with nilpotent variables.

9 Conclusions

Bosonic Logarithmic CFT was studied from a more geometric point of view, yielding familiar results of the definition of a logarithmic primary field. In particular, an example was found of a two-point function, where the two Jordan blocks were of different size, that was not set to zero by the global conformal invariance. The generators of the infinitesimal transformation were shown to integrate up to the geometric field defined. Using the machinery developed, the two-point function was obtained. The construction was applied to the $N = 1$ case, where again familiar results were found. The construction was then applied to the $N = 2$ case, and the two point function calculated. The only logarithmic divergences occurred in a manner familiar to the bosonic and $N = 1$ cases, even though the Cartan subalgebra is enlargened, and contains more non-diagonalizable elements than the bosonic and $N = 1$ cases.

Despite the fact that considering $\log dz$ has been useful in constructing LCFTs, precisely what it is is still not obvious to the author. It does not seem like something that could exist on a Riemann Sphere, perhaps some kind of covering of the sphere is

needed. What is still an intriguing question is - when demanding this type of non-unitary behaviour, what are the implications for the geometry of the underlying space on which the CFT is built?

Precisely how to generalize the machinery found in this note to $N = 3$ superconformal theories is not entirely obvious. The R-symmetry at the lie algebra level is given by $su(2)$, and is non-abelian. Presumably, representations of $su(2)$ where J_3 is non-diagonalizable would be required.

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